

LOWER BOUNDS FOR MOMENTS OF $\zeta'(\rho)$

MICAH B. MILINOVICH AND NATHAN NG

ABSTRACT. Assuming the Riemann Hypothesis, we establish lower bounds for moments of the derivative of the Riemann zeta-function averaged over the non-trivial zeros of $\zeta(s)$. Our proof is based upon a recent method of Rudnick and Soundararajan that provides analogous bounds for moments of L -functions at the central point, averaged over families.

1. INTRODUCTION

Let $\zeta(s)$ denote the Riemann zeta-function. In this article we are interested in obtaining lower bounds for moments of the form

$$J_k(T) = \frac{1}{N(T)} \sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{2k} \quad (1)$$

where $k \in \mathbb{N}$ and the sum runs over the non-trivial (complex) zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. As usual, we let the function

$$N(T) = \sum_{0 < \gamma \leq T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \quad (2)$$

denote the number of zeros of $\zeta(s)$ up to a height T counted with multiplicity.

Independently, Gonek [3] and Hejhal [5] have conjectured that $J_k(T) \asymp (\log T)^{k(k+2)}$ for each $k \in \mathbb{R}$. By modeling the Riemann zeta-function and its derivative using characteristic polynomials of random matrices, Hughes, Keating, and O'Connell [6] have refined this conjecture to state that $J_k(T) \sim C_k (\log T)^{k(k+2)}$ for a precise constant C_k when $k \in \mathbb{C}$ and $\Re k > -3/2$. However, we no longer believe this conjecture to be true for $\Re k < -3/2$. This is since we expect there exist infinitely many zeros ρ such that $|\zeta'(\rho)|^{-1} \gg |\gamma|^{1/3-\varepsilon}$ for each $\varepsilon > 0$.

Results of the sort suggested by these conjectures are only known for a few small values of k . See, for instance, the results of Gonek [1] for the case $k = 1$ and Ng [8] for the case $k = 2$. Also, Gonek [3] obtained a lower bound in the case $k = -1$. Our main result is to obtain a lower bound for $J_k(T)$ for each $k \in \mathbb{N}$ of the order of magnitude that is suggested by these conjectures.

Theorem 1. *Assume the Riemann Hypothesis and let $k \in \mathbb{N}$. Then for sufficiently large T we have*

$$\frac{1}{N(T)} \sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{2k} \gg_k (\log T)^{k(k+2)}.$$

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Under the assumption of the Riemann Hypothesis, Milinovich [7] has recently shown that $J_k(T) \ll_{k,\varepsilon} (\log T)^{k(k+2)+\varepsilon}$ for $k \in \mathbb{N}$ and $\varepsilon > 0$ arbitrary. When combined with Theorem 1, this result lends strong support for the conjecture of Gonek and Hejhal for k a positive integer.

Theorem 1 can be used to exhibit large values of $\zeta'(\rho)$. For example, as an immediate corollary we have the following result.

Corollary 1.1. *Assume the Riemann Hypothesis and let $\rho = \frac{1}{2} + i\gamma$ denote a non-trivial zero of $\zeta(s)$. Then for each $A > 0$ the inequality*

$$|\zeta'(\rho)| \geq (\log |\gamma|)^A \quad (3)$$

is satisfied infinitely often.

This result was previously proven by Ng [10] by an application of Soundararajan's resonance method [13]. The present proof is simpler and provides many more zeros ρ such that (3) is true. On the other hand, the resonance method is capable of detecting much larger values of $\zeta'(\rho)$ assuming a very weak form of the generalized Riemann hypothesis.

Our proof of Theorem 1 relies on combining a method of Rudnick and Soundararajan [11, 12] with a mean-value theorem of Ng (our Lemma 2) and a well-known lemma of Gonek (our Lemma 3). It is likely that our proof can be adapted to prove a lower bound for $J_k(T)$ of the conjectured order of magnitude for all rational k (with $k \geq 1$) in a manner analogous to that suggested in [11].

Let $k \in \mathbb{N}$ and define, for $\xi \geq 1$, the function $\mathcal{A}_\xi(s) = \sum_{n \leq \xi} n^{-s}$. Assuming the Riemann Hypothesis, we will estimate

$$\Sigma_1 = \sum_{0 < \gamma \leq T} \zeta'(\rho) \mathcal{A}_\xi(\rho)^{k-1} \overline{\mathcal{A}_\xi(\rho)}^k \quad \text{and} \quad \Sigma_2 = \sum_{0 < \gamma \leq T} |\mathcal{A}_\xi(\rho)|^{2k}$$

where the sums run over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. Hölder's inequality implies that

$$\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{2k} \geq \frac{|\Sigma_1|^{2k}}{(\Sigma_2)^{2k-1}},$$

and so we see that Theorem 1 will follow from the estimates

$$\Sigma_1 \gg T(\log T)^{k^2+2} \quad \text{and} \quad \Sigma_2 \ll T(\log T)^{k^2+1}. \quad (4)$$

It is convenient to express Σ_1 and Σ_2 slightly differently. Assuming the Riemann Hypothesis, $1 - \rho = \bar{\rho}$ for any non-trivial zero ρ of $\zeta(s)$. Thus, $\overline{\mathcal{A}_\xi(\rho)} = \mathcal{A}_\xi(1 - \rho)$. This allows us to re-write the sums in (1) as

$$\Sigma_1 = \sum_{0 < \gamma \leq T} \zeta'(\rho) \mathcal{A}_\xi(\rho)^{k-1} \mathcal{A}_\xi(1 - \rho)^k \quad \text{and} \quad \Sigma_2 = \sum_{0 < \gamma \leq T} \mathcal{A}_\xi(\rho)^k \mathcal{A}_\xi(1 - \rho)^k. \quad (5)$$

It is with these representations of Σ_1 and Σ_2 that we establish the bounds in (4).

2. SOME PRELIMINARY ESTIMATES

For each real number $\xi \geq 1$ and each $k \in \mathbb{N}$, we define the arithmetic sequence of real numbers $\tau_k(n; \xi)$ by

$$\sum_{n \leq \xi^k} \frac{\tau_k(n; \xi)}{n^s} = \left(\sum_{n \leq \xi} \frac{1}{n^s} \right)^k = \mathcal{A}_\xi(s)^k. \quad (6)$$

The function $\tau_k(n; \xi)$ is a truncated approximation to the arithmetic function $\tau_k(n)$ (the k -th iterated divisor function) which is defined by

$$\zeta^k(s) = \left(\sum_{n=1}^{\infty} \frac{1}{n^s} \right)^k = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s} \quad (7)$$

for $\Re s > 1$. We require a few estimates for sums involving the functions $\tau_k(n)$ and $\tau_k(n; \xi)$ in order to establish the bounds for Σ_1 and Σ_2 in (4).

We use repeatedly that, for $x \geq 3$ and $k, \ell \in \mathbb{N}$,

$$\sum_{n \leq x} \frac{\tau_k(n) \tau_\ell(n)}{n} \asymp_{k, \ell} (\log x)^{k\ell} \quad (8)$$

where the implied constants depend on k and ℓ . These bounds are well-known.

From (6) and (7) we notice that $\tau_k(n; \xi)$ is non-negative and $\tau_k(n; \xi) \leq \tau_k(n)$ with equality holding when $n \leq \xi$. In particular, choosing $k = \ell$ in (8) we find that, for $\xi \geq 3$,

$$(\log \xi)^{k^2} \ll_k \sum_{n \leq \xi} \frac{\tau_k(n)^2}{n} \leq \sum_{n \leq \xi^k} \frac{\tau_k(n; \xi)^2}{n} \leq \sum_{n \leq \xi^k} \frac{\tau_k(n)^2}{n} \ll_k (\log \xi)^{k^2}. \quad (9)$$

3. A LOWER BOUND FOR Σ_1

In order to establish a lower bound for Σ_1 , we require a mean-value estimate for sums of the form

$$S(X, Y; T) = \sum_{0 < \gamma \leq T} \zeta'(\rho) X(\rho) Y(1 - \rho)$$

where

$$X(s) = \sum_{n \leq N} \frac{x_n}{n^s} \quad \text{and} \quad Y(s) = \sum_{n \leq N} \frac{y_n}{n^s}$$

are Dirichlet polynomials. For $X(s)$ and $Y(s)$ satisfying certain reasonable conditions, a general formula for $S(X, Y; T)$ has been established by the second author [9]. Before stating the formula, we first introduce some notation. For T large, we let $\mathcal{L} = \log \frac{T}{2\pi}$ and $N = T^\vartheta$ for some fixed $\vartheta \geq 0$. The functions $\mu(\cdot)$ and $\Lambda(\cdot)$ are used to denote the usual arithmetic functions of Möbius and von Mangoldt. Also, we define the arithmetic function $\Lambda_2(\cdot)$ by $\Lambda_2(n) = (\mu * \log^2)(n)$ for each $n \in \mathbb{N}$.

Lemma 2. *Let x_n and y_n satisfy $|x_n|, |y_n| \ll \tau_\ell(n)$ for some $\ell \in \mathbb{N}$ and assume that $0 < \vartheta < 1/2$. Then for any $A > 0$, any $\varepsilon > 0$, and sufficiently large T we have*

$$\begin{aligned} S(X, Y; T) &= \frac{T}{2\pi} \sum_{mn \leq N} \frac{x_m y_{mn}}{mn} \left(\mathcal{P}_2(\mathcal{L}) - 2\mathcal{P}_1(\mathcal{L}) \log n + (\Lambda * \log)(n) \right) \\ &\quad - \frac{T}{4\pi} \sum_{mn \leq N} \frac{y_m x_{mn}}{mn} \mathcal{Q}_2(\mathcal{L} - \log n) + \frac{T}{2\pi} \sum_{\substack{a, b \leq N \\ (a, b) = 1}} \frac{r(a; b)}{ab} \sum_{g \leq \min\left(\frac{N}{a}, \frac{N}{b}\right)} \frac{y_{ag} x_{bg}}{g} \\ &\quad + O_A(T(\log T)^{-A} + T^{3/4 + \vartheta/2 + \varepsilon}) \end{aligned}$$

where $\mathcal{P}_1, \mathcal{P}_2$, and \mathcal{Q}_2 are monic polynomials of degrees 1, 2, and 2, respectively, and for $a, b \in \mathbb{N}$ the function $r(a; b)$ satisfies the bound

$$|r(a; b)| \ll \Lambda_2(a) + (\log T) \Lambda(a). \quad (10)$$

Proof. This is a special case of Theorem 1.3 of Ng [9]. \square

Letting $\xi = T^{1/(4k)}$, we find that the choices $X(s) = \mathcal{A}_\xi(s)^{k-1}$ and $Y(s) = \mathcal{A}_\xi(s)^k$ satisfy the conditions of Lemma 2 with $\vartheta = 1/4$, $N = \xi^k$, $x_n = \tau_{k-1}(n; \xi)$, and $y_n = \tau_k(n; \xi)$. Consequently, for this choice of ξ ,

$$\begin{aligned} \Sigma_1 &= \frac{T}{2\pi} \sum_{\substack{mn \leq \xi^k \\ m \leq \xi^{k-1}}} \frac{\tau_{k-1}(m; \xi) \tau_k(mn; \xi)}{mn} \left(\mathcal{P}_2(\mathcal{L}) - 2\mathcal{P}_1(\mathcal{L}) \log n + (\Lambda * \log)(n) \right) \\ &\quad - \frac{T}{4\pi} \sum_{mn \leq \xi^{k-1}} \frac{\tau_k(m; \xi) \tau_{k-1}(mn; \xi)}{mn} \mathcal{Q}_2(\mathcal{L} - \log n) \\ &\quad + \frac{T}{2\pi} \sum_{\substack{a, b \leq \xi^k \\ (a, b) = 1}} \frac{r(a; b)}{ab} \sum_{g \leq \min\left(\frac{N}{a}, \frac{N}{b}\right)} \frac{\tau_k(ag; \xi) \tau_{k-1}(bg; \xi)}{g} + O(T) \\ &= \mathcal{S}_{11} + \mathcal{S}_{12} + \mathcal{S}_{13} + O(T), \end{aligned}$$

say. To estimate \mathcal{S}_{11} , notice that, for T sufficiently large, $n \leq \xi^k = T^{1/4}$ implies that

$$\left(\mathcal{P}_2(\mathcal{L}) - 2\mathcal{P}_1(\mathcal{L}) \log n + (\Lambda * \log)(n) \right) \gg \mathcal{L}^2$$

and moreover, by (9),

$$\sum_{\substack{mn \leq \xi^k \\ m \leq \xi^{k-1}}} \frac{\tau_{k-1}(m; \xi) \tau_k(mn; \xi)}{mn} \geq \sum_{n \leq \xi^k} \frac{\tau_k(n; \xi)^2}{n} \gg (\log T)^{k^2}.$$

Thus, $\mathcal{S}_{11} \gg T(\log T)^{k^2+2}$. Since $\mathcal{Q}_2(\mathcal{L} - \log n) \ll \mathcal{L}^2$, we can bound \mathcal{S}_{12} by using the inequalities $\tau_k(n; \xi) \leq \tau_k(n)$ and $\tau_k(mn) \leq \tau_k(m) \tau_k(n)$. In particular, by twice using (8), we find that

$$\begin{aligned} \mathcal{S}_{12} &\ll T \mathcal{L}^2 \sum_{mn \leq \xi^k} \frac{\tau_k(m) \tau_{k-1}(m) \tau_k(n)}{mn} \leq T \mathcal{L}^2 \left(\sum_{m \leq T} \frac{\tau_k(m) \tau_{k-1}(m)}{m} \right) \left(\sum_{n \leq T} \frac{\tau_{k-1}(n)}{n} \right) \\ &\ll T(\log T)^{2+k(k-1)+k-1} \ll T(\log T)^{k^2+1}. \end{aligned}$$

It remains to consider the contribution from \mathcal{S}_{13} . Again using the inequalities $\tau_k(n; \xi) \leq \tau_k(n)$ and $\tau_k(mn) \leq \tau_k(m)\tau_k(n)$ along with (10), it follows that \mathcal{S}_{13} is bounded by

$$\begin{aligned} & \sum_{a, b \leq \xi^k} \frac{(\Lambda_2(a) + (\log T)\Lambda(a))}{ab} \sum_{g \leq \xi^k} \frac{\tau_k(a)\tau_k(g)\tau_{k-1}(b)\tau_{k-1}(g)}{g} \\ & \ll \sum_{a \leq T} \frac{(\Lambda_2(a) + (\log T)\Lambda(a))\tau_k(a)}{a} \sum_{b \leq T} \frac{\tau_{k-1}(b)}{b} \sum_{g \leq T} \frac{\tau_k(g)\tau_{k-1}(g)}{g} \\ & \ll (\log T)^{2+(k-1)+k(k-1)} = (\log T)^{k^2+1}. \end{aligned}$$

Combining this with our estimates for \mathcal{S}_{11} and \mathcal{S}_{12} , we conclude that $\Sigma_1 \gg T(\log T)^{k^2+2}$.

4. AN UPPER BOUND FOR Σ_2

Assuming the Riemann Hypothesis, we interchange the sums in (5) and find that

$$\Sigma_2 = N(T) \sum_{n \leq \xi^k} \frac{\tau_k(n; \xi)^2}{n} + 2\Re \sum_{m \leq \xi^k} \sum_{m < n \leq \xi^k} \frac{\tau_k(m; \xi)\tau_k(n; \xi)}{n} \sum_{0 < \gamma \leq T} \left(\frac{n}{m}\right)^\rho \quad (11)$$

where $N(T)$ denotes the number of non-trivial zeros of $\zeta(s)$ up to a height T . Recalling that $\xi = T^{1/(4k)}$ and using (2) and (9), it follows that

$$N(T) \sum_{n \leq \xi^k} \frac{\tau_k(n; \xi)^2}{n} \ll T(\log T)^{k^2+1}. \quad (12)$$

In order to bound the second sum on the right-hand side of (11), we require the following version of the Landau-Gonek explicit formula.

Lemma 3. *Let $x, T > 1$ and let $\rho = \beta + i\gamma$ denote a non-trivial zero of $\zeta(s)$. Then*

$$\begin{aligned} \sum_{0 < \gamma \leq T} x^\rho &= -\frac{T}{2\pi} \Lambda(x) + O(x \log(2xT) \log \log(3x)) \\ &\quad + O\left(\log x \min\left(T, \frac{x}{\langle x \rangle}\right)\right) + O\left(\log(2T) \min\left(T, \frac{1}{\log x}\right)\right) \end{aligned}$$

where $\langle x \rangle$ denotes the distance from x to the closest prime power other than x itself and $\Lambda(x) = \log p$ if x is a positive integral power of a prime p and $\Lambda(x) = 0$ otherwise.

Proof. This is a result of Gonek [2, 4]. □

Applying the lemma, we find that

$$\begin{aligned}
\sum_{m \leq \xi^k} \sum_{m < n \leq \xi^k} \frac{\tau_k(m; \xi) \tau_k(n; \xi)}{n} \sum_{0 < \gamma \leq T} \left(\frac{n}{m} \right)^\rho &= -\frac{T}{2\pi} \sum_{m \leq \xi^k} \sum_{m < n \leq \xi^k} \frac{\tau_k(m; \xi) \tau_k(n; \xi) \Lambda(\frac{n}{m})}{n} \\
&+ O \left(\mathcal{L} \log \mathcal{L} \sum_{m \leq \xi^k} \sum_{m < n \leq \xi^k} \frac{\tau_k(m; \xi) \tau_k(n; \xi)}{m} \right) \\
&+ O \left(\sum_{m \leq \xi^k} \sum_{m < n \leq \xi^k} \frac{\tau_k(m; \xi) \tau_k(n; \xi)}{m} \frac{\log \frac{n}{m}}{\langle \frac{n}{m} \rangle} \right) \\
&+ O \left(\log T \sum_{m \leq \xi^k} \sum_{m < n \leq \xi^k} \frac{\tau_k(m; \xi) \tau_k(n; \xi)}{n \log \frac{n}{m}} \right) \\
&= \mathcal{S}_{21} + \mathcal{S}_{22} + \mathcal{S}_{23} + \mathcal{S}_{24},
\end{aligned}$$

say. Since we only require an upper bound for Σ_2 (which, by definition, is clearly positive), we can ignore the contribution from \mathcal{S}_{21} because all the non-zero terms in the sum are negative. In what follows, we use ε to denote a small positive constant which may be different at each occurrence. To estimate \mathcal{S}_{22} , we note that $\tau_k(n; \xi) \leq \tau_k(n) \ll_\varepsilon n^\varepsilon$ which implies $\mathcal{S}_{22} \ll T^{1/4+\varepsilon}$. Turning to \mathcal{S}_{23} , we write n as $qm + \ell$ with $-\frac{m}{2} < \ell \leq \frac{m}{2}$ and find that

$$\mathcal{S}_{23} \ll T^\varepsilon \sum_{m \leq \xi^k} \frac{1}{m} \sum_{q \leq \lfloor \frac{\xi^k}{m} \rfloor + 1} \sum_{-\frac{m}{2} < \ell \leq \frac{m}{2}} \frac{1}{\langle q + \frac{\ell}{m} \rangle}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x . Notice that $\langle q + \frac{\ell}{m} \rangle = \frac{|\ell|}{m}$ if q is a prime power and $\ell \neq 0$, otherwise $\langle q + \frac{\ell}{m} \rangle$ is $\geq \frac{1}{2}$. Hence,

$$\begin{aligned}
\mathcal{S}_{23} &\ll T^\varepsilon \left(\sum_{m \leq \xi^k} \frac{1}{m} \sum_{\substack{q \leq \lfloor \frac{\xi^k}{m} \rfloor + 1 \\ \Lambda(q) \neq 0}} \sum_{1 \leq \ell \leq \frac{m}{2}} \frac{m}{\ell} + \sum_{m \leq \xi^k} \frac{1}{m} \sum_{q \leq \lfloor \frac{\xi^k}{m} \rfloor + 1} \sum_{1 \leq \ell \leq \frac{m}{2}} 1 \right) \\
&\ll T^\varepsilon \left(\sum_{m \leq \xi^k} \sum_{q \leq \lfloor \frac{\xi^k}{m} \rfloor + 1} 1 \right) \ll T^{1/4+\varepsilon}.
\end{aligned}$$

It remains to consider \mathcal{S}_{24} . For integers $1 \leq m < n \leq \xi^k$, let $n = m + \ell$. Then

$$\log \frac{n}{m} = -\log \left(1 - \frac{\ell}{m} \right) > \frac{\ell}{m}.$$

Consequently,

$$\mathcal{S}_{24} \ll T^\varepsilon \sum_{m \leq \xi^k} \sum_{1 \leq \ell \leq \xi^k} \frac{1}{(m + \ell) \frac{\ell}{m}} \ll T^\varepsilon \xi^k = T^{1/4+\varepsilon}. \quad (13)$$

Combining (12) with our estimates for \mathcal{S}_{22} , \mathcal{S}_{23} , and \mathcal{S}_{24} we deduce that $\Sigma_2 \ll T(\log T)^{k^2+1}$ which, when combined with our estimate for Σ_1 , completes the proof of Theorem 1.

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Micah B. Milinovich
 Math Department
 University of Rochester
 Rochester, NY
 14627 USA
 micah@math.rochester.edu

Nathan Ng
 Department of Mathematics and Statistics
 University of Ottawa
 585 King Edward Avenue
 Ottawa, ON
 K1N 6N5 Canada
 nng@uottawa.ca